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REDUCTION OF AN ELLIPTIC INTEGRAL TO LEGENDRE'S NORMAL FORM

BY NORMAN R. WILSON

IN his *Traité des fonctions elliptiques*, Legendre has shown that the integral $\int \frac{dx}{\sqrt{\psi(x)}}$ (x real and $\psi(x)$ a quartic in x with real coefficients) can be reduced to the normal form $\int \frac{d\varphi}{\sqrt{1-\kappa^2 \sin^2\varphi}}$ (κ real and less than 1) by a real transformation of the variables. The problem is considered under six heads. Cayley observes (*Elliptic Functions* § 414) that of these only five are distinct. The following note reduces the number to three, an invariantic method being used.* It is assumed that the roots of $\psi(x) = 0$ are all distinct, and further that x is restricted to an interval such that the expression under the radical is positive.

Writing the integral in the homogeneous form, it becomes

$$\int \frac{x_2 dx_1 - x_1 dx_2}{\sqrt{A_0 x_1^4 + 4A_1 x_1^3 x_2 + 6A_2 x_1^2 x_2^2 + 4A_3 x_1 x_2^3 + A_4 x_2^4}}.$$

Resolve the quartic in the denominator into the real quadratic factors, $f \equiv a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$ and $f' \equiv b_0 x_1^2 + 2b_1 x_1 x_2 + b_2 x_2^2$, and let the roots of $f = 0$ and $f' = 0$ be α_1, α_2 and β_1, β_2 respectively. If only two of these are real, they are to be β_1 and β_2, and if all are real, β_1 and β_2 are to be the two between† which x lies.

1. The sign of $\psi(x)$ and of the resultant of f and f'. If R denote the resultant, and the quartic have all its roots finite,

$$R = a_0^2 b_0^2 (\alpha_1 - \beta_1)(\alpha_1 - \beta_2)(\alpha_2 - \beta_1)(\alpha_2 - \beta_2).$$

(Weber, *Algebra* §53).

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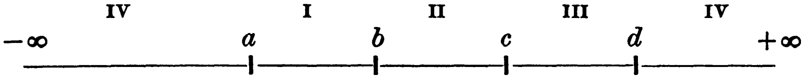
* The reduction can also be made, in a similar manner, by very elementary algebra.

† The word *"between"* is used in the sense of projective geometry. Thus if *abcd* be the roots in ascending order of magnitude, x would be said to lie between d and a if $x > d$ or $x < a$.

(a) *If* $\psi(x) = 0$ *has all its roots real and finite.* Since β_1, β_2 are the roots between which x lies, neither or both of these lie between α_1 and α_2. R will therefore be positive. Representing the values of the roots geometrically as a, b, c, d, and remembering that $\psi(x) \equiv A_0(x - \alpha_1)(x - \alpha_2)(x - \beta_1)(x - \beta_2)$,



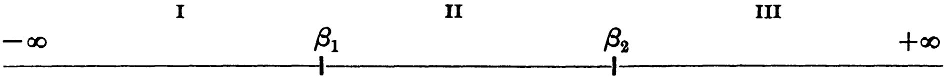
it is evident that when A_0 is positive, in order that $\psi(x)$ shall be positive, x must lie in the interval II or IV, and when A_0 is negative, in the interval I or III. Remembering that in each case β_1, β_2 are the extremities of these intervals, it is easily seen from the figures that the sign of R is positive.

(b) *If* $\psi(x) = 0$ *has complex, but no infinite roots.* When there are two real roots, β_1, β_2, $\alpha_1 - \beta_1$, $\alpha_2 - \beta_1$ and $\alpha_1 - \beta_2$, $\alpha_2 - \beta_2$ will be pairs of conjugate complexes; and when all the roots are complex $\alpha_1 - \beta_1$, $\alpha_2 - \beta_2$ and $\alpha_1 - \beta_2$, $\alpha_2 - \beta_1$ will be conjugate. Hence in both cases R will be positive.

(c) $\psi(x) = 0$ *has an infinite root.* In this case $A_0 = 0$. If the integral is to remain elliptic, a_0 and b_0 cannot both vanish. If $a_0 = 0$, all the roots of $\psi(x) = 0$ are real. Since x is to lie between β_1 and β_2, the finite root of $f = 0$ cannot lie between them. Writing f in the form $a(x - \alpha)$ and f' as $b(x - \beta_1)(x - \beta_2)$, and substituting for the coefficients in the expression for R (Weber, *Algebra*, § 53), we find

$$R = a^2b^2(\alpha - \beta_1)(\alpha - \beta_2),$$

which is positive since α must lie outside the interval $\beta_1\ \beta_2$.

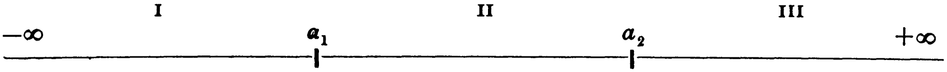


Now $\psi(x) \equiv ab(x - \alpha)(x - \beta_1)(x - \beta_2)$. If $\psi(x)$ is positive, ab and $x - \alpha$ must have different signs. Representing the values of the roots β_1, β_2 geometrically, x lies in the interval II. If ab be positive, α lies in III and if ab be negative, in I. In both cases R is positive.

On the other hand, if $b_0 = 0$, and we write $f \equiv a(x - \alpha_1)(x - \alpha_2)$ and $f \equiv b(x - \beta)$, then

$$R = a^2b^2(\alpha_1 - \beta)(\alpha_2 - \beta).$$

If α_1, α_2 are complex, this is positive. If they are real, β cannot lie between them. In the latter case, $x - \alpha_1$ and $x - \alpha_2$ have the same sign. Hence $(x - \beta)$ must have the same sign as ab.



If ab be positive, β and x must both lie in III, $\beta < x$; if ab be negative, β and x must both lie in I and $x < \beta$.

2. The Algebraic Reduction. The values of λ for which $f + \lambda f'$ is a perfect square are given by $(a_0 + \lambda b_0)(a_2 + \lambda b_2) - (a_1 + \lambda b_1)^2 = 0$, or

$$D''\lambda^2 + 2D'\lambda + D = 0 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (\alpha),$$

where $D \equiv 2(a_0 a_2 - a_1^2)$; $D'' = 2(b_0 b_2 - b_1^2)$; $D' = a_0 b_2 + a_2 b_0 - 2a_1 b_1$.

Now $D'^2 - DD''$ is easily seen to be equal to R, which we have shown to be positive (§1). Hence the roots λ_1 and λ_2 of (α) are real and unequal. The coefficient of x_1^2 in $f + \lambda f'$ being $(a_0 + \lambda b_0)$ we may write

$$\left.\begin{aligned} f + \lambda_1 f' &\equiv (a_0 + \lambda_1 b_0)(x_1 - \eta\, x_2)^2 \equiv \epsilon_1\, y_1^2, \\ f + \lambda_2 f' &\equiv (a_0 + \lambda_2 b_0)(x_1 - \theta\, x_2)^2 \equiv \epsilon_2\, y_2^2, \end{aligned}\right\} \cdot \cdot \cdot \cdot \cdot \cdot \cdot (\beta)$$

where ϵ_1, ϵ_2 stand for ± 1 according as $(a_0 + \lambda_1 b_0)$ and $(a_0 + \lambda_2 b_0)$ are respectively positive and negative. Hence x_1, x_2 and y_1, y_2 are connected by the linear substitution

$$\left.\begin{aligned} y_1 &= \pm\sqrt{\frac{a_0 + \lambda_1 b_0}{\epsilon_1}}\;(x_1 - \eta x_2), \\ y_2 &= \pm\sqrt{\frac{a_0 + \lambda_2 b_0}{\epsilon_2}}\;(x_1 - \theta x_2), \end{aligned}\right\} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (\gamma)$$

the signs of the square roots being still at our disposal. Solving (β) for f and f', the quartic ff' reduces to

$$\frac{-\lambda_1}{(\lambda_1 - \lambda_2)^2}\left(y_2^2 - \frac{\epsilon_1}{\epsilon_2} y_1^2\right)\left(y_2^2 - \frac{\epsilon_1}{\epsilon_2} \cdot \frac{\lambda_2}{\lambda_1} \cdot y_1^2\right) \cdot$$

Writing μ and ν for $\sqrt{(a_0 + \lambda_1 b_0)\epsilon_1}$ and $\sqrt{(a_0 + \lambda_2 b_0)\epsilon_2}$, we have

$$\mu\nu = \sqrt{\epsilon_1\epsilon_2(a_0 + \lambda_1 b_0)(a_0 + \lambda_2 b_0)} = \sqrt{\frac{-2\epsilon_1\epsilon_2}{D''}}\,(a_0b_1 - a_1b_0),$$

on substituting for $(\lambda_1 + \lambda_2)$ and $\lambda_1\lambda_2$ from (α), and performing simple reductions. The equation giving η and θ (the Jacobian; *cf.* Salmon, *Algebra*, §177) is

$$(a_0b_1 - a_1b_0)x_1^2 + (a_0b_2 - a_2b_0)x_1x_2 + (a_1b_2 - a_2b_1)x_2^2 = 0,$$

whence

$$\begin{aligned}(a_0b_1 - a_1b_0)^2(\eta - \theta)^2 &= (a_2b_0 - a_2b_0)^2 - 4(a_0b_1 - a_1b_0)(a_1b_2 - a_2b_1)\\ &= R.\end{aligned}$$

Now R has been shown to be positive in §1. Hence θ and η are real, and, therefore, the substitution (γ) is real. Further, $\mu\nu(\eta - \theta)$ reduces to $\sqrt{\frac{-2\epsilon_1\epsilon_2 R}{D''}}$. We choose the signs of μ and ν so that $\mu\nu(\eta - \theta)$ is equal to the positive value of this square root. Now

$$\begin{vmatrix} y_2 & dy_2 \\ y_1 & dy_1 \end{vmatrix} = \begin{vmatrix} \nu x_1 - \nu\theta x_2, & \nu dx_1 - \nu\theta dx_2 \\ \mu x_1 - \mu\eta x_2, & \mu dx_1 - \mu\eta dx_2 \end{vmatrix} = \begin{vmatrix} \nu, & -\nu\theta \\ \mu, & -\mu\eta \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 \\ dx_1 & dx_2 \end{vmatrix}.$$

Hence $(x_2\,dx_1 - x_1\,dx_2) = (y_2\,dy_1 - y_1\,dy_2) \div \mu\nu(\eta - \theta)$. Further, from ($\alpha$), we have

$$(\lambda_1 - \lambda_2)^2 = \frac{4(D'^2 - DD'')}{D''^2} = \frac{4R}{D''^2}.$$

Substituting in our integral, and returning to non-homogeneous variables, we have

$$\sqrt{\frac{2}{\lambda_1 \cdot D'' \cdot \epsilon_1\epsilon_2}} \int \frac{dy}{\sqrt{\left(1 - \frac{\epsilon_1}{\epsilon_2}\,y^2\right)\left(1 - \frac{\epsilon_1}{\epsilon_2}\frac{\lambda_2}{\lambda_1}\,y^2\right)}}\,.$$

Also, since the modulus, $\mu\nu(\theta - \eta)$ or $\sqrt{\frac{-2\epsilon_1\epsilon_2 R}{D''}}$, of the substitution ($\gamma$) must be real, $\epsilon_1\epsilon_2$ must have a sign opposite to that of D''. We consider the following cases:

(*a*) $\psi(x) = 0$ *has four real roots.* When the roots are also finite, D' must be a function of the differences, easily seen to be

$$\tfrac{1}{2} A_0 \{ (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_1 - \beta_2)(\alpha_2 - \beta_1) \}.$$

If A_0 be positive, a reference to the geometrical representation of § 1(*a*) will show that both terms within the bracket are negative. If A_0 be negative, a similar result follows. If $A_0 = 0$ and $b_0 = 0$ the expression for D' becomes $ab\{(\alpha_1 - \beta) + (\alpha_2 - \beta)\}$, which a reference to § 1(*c*) shows to be negative. If $a_0 = 0$, D' becomes $ab\{(\beta_1 - \alpha) + (\beta_2 - \alpha)\}$, which is again negative by § 1(*c*).

In all cases, therefore, D' is negative. Since $f = 0$ and $f' = 0$ have real roots, D and D'' are both negative. Hence both roots λ_1, λ_2 of (α) are negative. Let $|\lambda_1|$ be the greater. Then we may write $\lambda^2 = \frac{\lambda_2}{\lambda_1}$ (λ real and $|\lambda|$ less than 1). Since D'' is negative, $\epsilon_1\epsilon_2 = -1$, and our integral reduces to

$$\sqrt{\frac{2}{\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 - y^2)(1 - \lambda^2 y^2)}}, \qquad 0 < \lambda^2 < 1.$$

(*b*) $\psi(x) = 0$ *has two real and two complex roots.* In this case, D is positive and D'' negative. Hence (α) has a positive and a negative root, λ_2 and λ_1 respectively. Further $\epsilon_1\epsilon_2 = -1$ since D'' is negative. Writing $\frac{\lambda_2}{\lambda_1} = -\lambda^2$, we obtain our integral in the form

$$\sqrt{\frac{2}{\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 - y^2)(1 + \lambda^2 y^2)}}.$$

(*c*) $\psi(x) = 0$ *has four complex roots.* If $\psi(x) = 0$ is to be positive, A_0 must be positive. Hence D' or $A_0\{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_2 - \beta_1)(\alpha_1 - \beta_2)\}$, being the sum of two products of conjugate complex quantities, is positive. So are D and D''. Hence the two roots of (a) are negative. Choose $|\lambda_1| > |\lambda_2|$; since the roots are unequal, this may be done. Writing $\frac{\lambda_2}{\lambda_1} = \lambda^2$, we obtain

$$\sqrt{\frac{2}{-\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 + y^2)(1 + \lambda^2 y^2)}}, \qquad 0 < \lambda^2 < 1.$$

Hence we have reduced the integral to the form:

$$\text{real constant} \times \int \frac{dy}{\sqrt{(1 \pm y^2)(1 \pm \lambda^2 y^2)}},$$

where λ is real, and in the first and last cases $\lambda^2 < 1$.

3. The Trigonometric Reduction.

(*a*) $\psi(x) = 0$ *has four real roots.* f' has been transformed into $(1 - y^2)$ multiplied by a certain constant (see equations (β)). Hence the anharmonic ratio $(x, \beta_1, \alpha_1, \beta_2)$ becomes $(y, 1, \pm 1/\lambda, -1)$. The transforming substitution is

$$\frac{(x-\beta_1)}{(x-\beta_2)} \cdot \frac{(\alpha_1-\beta_2)}{(\alpha_1-\beta_2)} = \frac{(y-1)(1 \pm \lambda)}{(y+1)(1 \pm \lambda)} \; .$$

But $|\lambda| < 1$ (see § 1(*a*)). Referring to § 1(*a*), if $\beta_1 < \alpha_1 < \alpha_2 < \beta_2$, x must lie in the interval IV, and $\frac{x-\beta_1}{x-\beta_2}$ is positive, while $\frac{\alpha_1-\beta_1}{\alpha_1-\beta_2}$ is negative. If $\alpha_1 < \beta_1 < \beta_2 < \alpha_2$ x lies in the interval II, and $\frac{x-\beta_1}{x-\beta_2}$ is negative, while $\frac{\alpha_1-\beta_1}{\alpha_1-\beta_2}$ is positive. In both cases $\frac{y-1}{y+1}$ must be negative. If one of the roots of $\psi = 0$ is infinite, and it be a root of $f = 0$, then taking α_1 as the other root, the above argument is not affected. If it be a root of $f' = 0$, β_1 say, the anharmonic equality reduces to

$$\frac{\alpha_1-\beta_1}{\alpha_1-\beta_2} = \frac{(y-1)(1 \pm \lambda)}{(y+1)(1 \pm \lambda)} \, .$$

Now α_1 and x must be on different sides of β_2 (see § 1 (*c*)). Hence $\frac{\alpha_1-\beta}{x-\beta_2}$ is negative, and therefore the same is true of $\frac{y-1}{y+1}$. Hence in all cases $|y| < 1$. Write $\sin\varphi$ for y and κ for λ and we obtain

$$\sqrt{\frac{2}{\lambda_1 D''}} \int \frac{d\varphi}{\sqrt{1-\kappa^2 \sin^2\varphi}} \; ,$$

or, if it is desired to express it in terms of more familiar invariants,

$$\sqrt{\frac{2}{\sqrt{R_1} - D'}} \int \frac{d\varphi}{\sqrt{1-\kappa^2 \sin^2\varphi}} \; .$$

(*b*) $\psi(x) = 0$ *has two real and two complex roots.* Since $\sqrt{\frac{2}{\lambda_1 D''}}$ is real, and our integral real, $(1-y^2)$ must be positive, and therefore $|y|<1$. Put $\cos\varphi$ for y and we get

$$-\sqrt{\frac{2}{\lambda_1 D''}}\int\frac{d\varphi}{\sqrt{1+\lambda^2\cos^2\varphi}}=-\sqrt{\frac{2}{\lambda_1 D''(1+\lambda^2)}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}},$$

where $\kappa^2=\frac{\lambda^2}{1+\lambda^2}$, and is therefore positive and less than 1. Since $\lambda^2=-\frac{\lambda_2}{\lambda_1}$, this becomes

$$-\sqrt{\frac{2}{(\lambda_1-\lambda_2)D''}}=-\frac{1}{\sqrt[4]{R}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}},$$

where $\kappa^2=\frac{\lambda_2}{\lambda_2-\lambda_1}$ From the signs of D, D'', it is easily seen that $\lambda_1=\frac{-D'+\sqrt{R}}{D''}$, and hence $\kappa^2=\frac{D'+\sqrt{R}}{2\sqrt{R}}$.

(*c*) $\psi(x)=0$ *has four complex roots.* Put $y=\tan\varphi$, and our integral becomes

$$\sqrt{\frac{-2}{\lambda_1 D''}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}},$$

where $\kappa^2=1-\lambda^2$, and is therefore positive and less than 1 (see §2(*c*)). Hence, substituting for λ^2, we have $\kappa^2=\frac{\lambda_1-\lambda_2}{\lambda_1}$, and by (*a*) we find $\lambda_1=\frac{-D-\sqrt{R}}{D''}$ (see §2 (*c*)). Hence we obtain

$$\sqrt{\frac{2}{D'+\sqrt{R}}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}.$$

We may summarize as follows: To reduce to Legendre's normal form the integral $\int\frac{dx}{\sqrt{\psi(x)}}$ (x real and $\psi(x)$ a quartic in x with real coefficients and unequal roots), resolve the quartic into real quadratic factors, f and f', where, if only two of the roots of $\psi(x)=0$ are real, $f'=0$ contains them, and if they are all real, $f'=0$ contains the pair between which x lies. Let $f\equiv a_0x^2+2a_1x+a_2$, and $f'\equiv b_0x^2+2b_1x+b_2$. Find the invariants

$$D\equiv 2(a_0a_2-a_1^2),\ D''\equiv 2(b_0b_2-b_1^2),\ D'=a_0b_2+a_2b_0-2a_1b_1,$$
$$R\equiv(a_0b_2-a_2b_0)^2-4(a_0b_1-a_1b_0)(a_1b_2-a_2b_1).$$

Find also the roots, necessarily real and unequal, of the quadratic equation

$$D''\lambda^2 + 2D'\lambda + D = 0.$$

Call the lesser of these λ_1 and the greater λ_2. Then the normal form, and the transforming substitution are given in the following table, where in every case $0 < \kappa^2 < 1$.

CASE.	NORMAL FORM.	κ^2	κ^2	TRANSFORMATION.
Four real roots or $D -\quad ; D'' -$	$\sqrt{\frac{2}{\sqrt{R}-D'}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$	$\frac{\lambda_2}{\lambda_1}$	$\frac{R+D'^2+2D'\sqrt{R}}{DD''}$	$\pm\sqrt{\frac{f+\lambda_1 f'}{f+\lambda_2 f'}} = \sin\varphi$
Two real roots, two complex roots, or $D +\quad ; D'' -$	$\frac{-1}{\sqrt[4]{R}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$	$\frac{\lambda_2}{\lambda_2-\lambda_1}$	$\frac{D'+\sqrt{R}}{2\sqrt{R}}$	$\pm\sqrt{\frac{f+\lambda_1 f'}{f+\lambda_2 f'}} = \cos\varphi$
Four complex roots or $D +\quad ; D'' +$	$\sqrt{\frac{2}{D'+\sqrt{R}}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$	$\frac{\lambda_1-\lambda_2}{\lambda_1}$	$\frac{2\sqrt{R}}{\sqrt{R}+D'}$	$\pm\sqrt{-\frac{f+\lambda_1 f'}{f+\lambda_2 f'}} = \tan\varphi$

By $\sqrt{\ }$ is understood the positive square root. In order that the given and the reduced integral may agree in sign, it is necessary that in extracting the root given in the last column of this table that sign shall be taken which renders $(ad - bc)$ positive, $\frac{ax+b}{cx+d}$ being the root.

Chicago, August, 1903.